

Instability of a conducting fluid slab carrying uniform current in the presence of a homogeneous magnetic field

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With 2 figures in the text

ABSTRACT

The stability of an infinite slab of an incompressible, inviscid fluid of low electrical conductivity carrying a uniform current produced by external sources is investigated in two cases: (1) when both the surfaces of the fluid slab are free and (2) when one surface is free, the other being rigid. In the latter case the influence of gravity acting perpendicular to the surface of the slab is included. The influence of surface tension of the fluid is, however, taken into account in both the cases. The effect of a homogeneous magnetic field produced by external sources and directed along the current in the fluid is also discussed.

It is shown that the fluid slab is always unstable when it carries an electric current which is closed outside the fluid symmetrically at infinity. In particular when the slab has both of its surfaces free the fundamental modes are either symmetric or anti-symmetric about the mid-plane in the absence of external magnetic field while they are neither symmetric nor anti-symmetric in its presence.

1. Introduction

Kruskal and Schwarzschild [1] studied among other problems the stability of a compressible fluid slab of infinite electrical conductivity with volume as well as surface current. The fluid is assumed to have the magnetic field directed everywhere perpendicular to the current and extending outside the fluid also. Assuming the strength of the field outside the fluid to be larger than the inside value, they find that the fluid slab is stable to sinusoidal perturbations of the boundaries. A similar problem has also been studied by Loughhead [2], who showed that an incompressible fluid slab can be unstable for some wavelengths if the magnetic field outside the fluid has a smaller value than inside.

In the study of the stability of the fluid slab the above-mentioned authors assumed that the fluid had infinite electrical conductivity. This assumption is good in the case of a plasma and under the conditions one finds in astrophysical problems. In these situations one rarely comes across a geometric configuration which can be approximated to a slab of finite thickness and extending to

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infinity in the other directions. However, one can realise in the laboratory, using mercury or liquid sodium a configuration approximating a flat slab of fluid by forming a tank with depth small compared with the other linear dimensions or by a jet in the form of a sheet. Both these liquids have large conductivity but the characteristic dimensions of the problem are of the order of 1 cm and therefore the assumption of infinite electrical conductivity is not justified. On the other hand, one can obtain the hydromagnetic stability criteria for fluids whose electrical conductivity is of the order that of mercury in the limit of zero electrical conductivity if the currents are maintained by external surces [3, 4]. It is the aim of this paper to study the stability of a fluid slab carrying uniform volume current in the presence of a homogeneous external magnetic field directed along the current under the assumption of low electrical conductivity of the fluid.

This problem will be studied in two parts. In the first part the stability of a fluid slab with one surface rigid and the other free, taking into account the effect of acceleration due to gravity acting perpendicular to the surface of the fluid, will be investigated. In the second part the stability of the fluid slab with both surfaces free, neglecting the effect of gravity, will be investigated. The influence of surface tension will be included in both the cases.

PART I

Instability of a fluid slab with one surface free and the other rigid

2. Formulation of the problem

Let us consider a slab of an incompressible, inviscid and non-magnetic fluid confined between two parallel planes separated by a distance $2d$. Let us choose a right-handed Cartesian coordinate system with the z -axis perpendicular to the surface of the fluid with the origin situated midway between the planes. Then the two planes of the fluid slab are given by $z = \pm d$. The fluid is assumed to extend to infinity in x and y directions. Let there be an uniform current of constant volume density, j_0 , flowing in the y direction inside the fluid slab and closed at infinity symmetrically on either side of the slab. The solution of the problem when the current is not symmetrically closed offers no difficulties, but we shall confine ourselves to the above assumption in the present paper. A uniform magnetic field, B_u , produced by external sources is also present and is directed along the y -axis. We shall assume that the plane $z = -d$ is rigid, while the plane $z = d$ is free. The force of gravity is assumed to be acting in the negative z direction (see Fig. 1). We shall further assume that the regions $z > d$ and $z < -d$ are vacuums.

The magnetic field distribution due to the current is given in rationalised M.K.S. units by

$$\mathbf{B} = (B_x, 0, 0), \quad (2.1 a)$$

$$\text{where } B_x = \mu_0 j_0 z, \quad \text{if } -d \leq z \leq +d, \quad (2.1 b)$$

$$B_x = \mu_0 j_0 d, \quad \text{if } z \geq d, \quad (2.1 c)$$

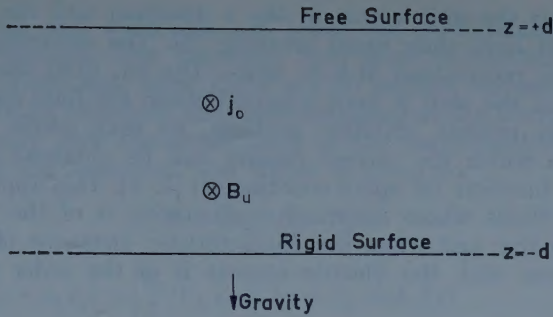


Fig. 1. Geometric configuration of a fluid slab confined between two planes ($z = d$ and $z = -d$) of which the plane $z = -d$ is rigid and the plane $z = d$ is free. The fluid extends to infinity in other two directions. A current of constant volume density is flowing in the direction of an externally applied homogeneous magnetic field which is parallel to the y -axis.

$$\text{and} \quad B_x = -\mu_0 j_0 d, \quad \text{if } z \leq -d. \quad (2.1 d)$$

The fluid slab will be in equilibrium with pressure distribution given inside the fluid by

$$p = \frac{1}{2} \mu_0 j_0^2 (d^2 - z^2) - \rho g (z - d), \quad (2.2)$$

where ρ = density of the fluid and g = acceleration due to gravity.

In order to study the stability of the fluid slab we assume that the free surface of the fluid is perturbed and its shape is then given by

$$z = d + a \cos(k_x x + k_y y), \quad (2.3)$$

where $|a| \ll d$. The signs of k_x and k_y are for the present kept arbitrary. We assume further that a is periodic in time such that

$$a = a_0 \cos \omega t. \quad (2.4)$$

The purpose of the theory is to obtain the dispersion relation between ω and k_x and k_y , which enables us to find out the range of values of k_x and k_y for which ω has an imaginary value which corresponds to unstable modes of the perturbation.

The basic equations of the problem are

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} - \rho g \hat{\mathbf{z}}, \quad (2.5)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.6)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad (2.7)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.8)$$

$$I = \text{constant}. \quad (2.9)$$

In eq. (2.5) $\hat{z}$ is the unit vector in the z direction and the other symbols in eqs. (2.5) to (2.8) have their usual meaning [5]. The electromagnetic quantities are expressed in rationalised M.K.S. units. The eq. (2.9) shows that the total current flowing in the slab is kept constant when the fluid surface is perturbed.

In solving the present stability problem we shall adopt the magnetostatic approximation in which the current density can be obtained as a gradient of a harmonic scalar function (of space coordinates) [3, 4]. This approximation is good in the case of fluids whose electrical conductivity is of the order of the conductivity of mercury and when the characteristic distances of the problem are of order of 1 cm and the electric current is of the order of a few hundred amperes.

3. Solution for ω

(a) Current density distribution in the magnetostatic approximation

As stated above, the current density is the gradient of a harmonic function. This fact together with eq. (2.9) and the boundary conditions that there should be no component of current density vector normal to the boundary given by eq. (2.3) and also at the boundary $z = -d$ enables one to determine uniquely the current density distribution inside the perturbed fluid slab. In the calculation of any physical quantity we shall retain only terms proportional to a/d and neglect higher order terms. Following the method adopted in [3, 4] it can be shown that the perturbed current density is given by

$$\mathbf{j} = (j_x, j_y, j_z), \quad (3.1)$$

$$\text{where} \quad j_x = k_x [A_1 \cosh kz + A_2 \sinh kz] \cos \varphi, \quad (3.2 a)$$

$$j_y = j_0 + k_y [A_1 \cosh kz + A_2 \sinh kz] \cos \varphi, \quad (3.2 b)$$

$$\text{and} \quad j_z = k [A_1 \sinh kz + A_2 \cosh kz] \sin \varphi, \quad (3.2 c)$$

where A_1 and A_2 are constants given by

$$A_1 = -\frac{a j_0 k_y}{2k \sinh kd}, \quad (3.3)$$

$$\text{and} \quad A_2 = -\frac{a j_0 k_y}{2k \cosh kd}, \quad (3.4)$$

$$\text{where} \quad k \equiv (k_x^2 + k_y^2)^{\frac{1}{2}}, \quad (3.5)$$

$$\text{and} \quad \varphi \equiv (k_x x + k_y y). \quad (3.6)$$

(b) Magnetic field distribution

The distribution of the magnetic field inside the fluid slab is determined by eqs. (2.7) and (2.8). In the region outside the fluid, the magnetic field vector

is both irrotational and solenoidal and hence can be obtained as the gradient of a harmonic function. The magnetic field must be continuous at both the boundaries of the fluid slab since there are no surface currents. The method of obtaining the relevant solutions and their matching at the boundaries is straightforward and we only give the distribution of the magnetic field, B_i , inside the fluid in which we are interested and which is given by

$$\mathbf{B}_i = (B_{xi}, B_{yi}, B_{zi}), \quad (3.7 a)$$

where

$$B_{xi} = \mu_0 j_0 z + \left\{ \frac{k_x}{k} (C_1 \sinh kz + C_2 \cosh kz) + \frac{\mu_0 k_y}{k} (A_1 \sinh kz + A_2 \cosh kz) \right\} \cos \varphi, \quad (3.7 b)$$

$$B_{yi} = \left\{ \frac{k_y}{k} (C_1 \sinh kz + C_2 \cosh kz) - \frac{\mu_0 k_x}{k} (A_1 \sinh kz + A_2 \cosh kz) \right\} \cos \varphi, \quad (3.7 c)$$

and

$$B_{zi} = (C_1 \cosh kz + C_2 \sinh kz) \sin \varphi, \quad (3.7 d)$$

where C_1 and C_2 are constants given by

$$C_1 = C_2 = -\frac{1}{2k} \mu_0 a j_0 k_x \exp(-kd). \quad (3.8)$$

(c) The pressure distribution

It has been shown by Murty [3, 4] that the pressure distribution inside the fluid can be obtained from the solution of a Poisson's equation with the right-hand side given by the current density alone when the current density vector is a gradient of a scalar function and if there are no other forces acting on the fluid. However, the presence of a volume force due to gravity does not introduce any difficulty if the sources of the force are outside the fluid (i.e. if we neglect the self-gravity of the fluid). This can be seen as follows. If we take the divergence of both sides of eq. (2.5) and use eq. (2.6), we obtain

$$\nabla^2 p = -\mu_0 j^2, \quad (3.9)$$

where we have used the condition that $\mathbf{j}$ is the gradient of a harmonic function. Using eqs. (3.1), (3.2) and (3.9) it can be shown that the pressure, p , inside the fluid slab is given by

$$p = E_1 + E_2 z - \frac{1}{2} \mu_0 j_0^2 z^2 + \left\{ D_1 \cosh kz + D_2 \sinh kz - \frac{\mu_0 k_y j_0 z}{k} (A_1 \sinh kz + A_2 \cosh kz) \right\} \cos \varphi, \quad (3.10)$$

where E_1 and E_2 are constants of order unity and D_1 and D_2 are constants of order a/d . The boundary conditions for the determination of these constants are: (1) the pressure at a point just inside the perturbed boundary is that due to surface tension and (2) the z -component of the total force density must be zero at the boundary $z = -d$. Using these boundary conditions one can show that

$$E_1 = \frac{1}{2} \mu_0 j_0^2 d^2 - E_2 d, \quad (3.11 a)$$

$$E_2 = -\rho g, \quad (3.11 b)$$

$$D_1 = \frac{1}{\cosh 2kd} [S_1 \cosh kd - S_2 \sinh kd], \quad (3.11 c)$$

$$D_2 = \frac{1}{\cosh 2kd} [S_1 \sinh kd + S_2 \cosh kd], \quad (3.11 d)$$

where
$$S_1 = a [T k^2 + \rho g + \mu_0 d j_0^2 k_x^2 / k^2], \quad (3.12 a)$$

and
$$S_2 = \frac{a}{2k^3} \left[\mu_0 j_0^2 k_x^2 e^{-2kd} - \frac{2k k_x k_y j_0 B_u}{\sinh 2kd} \right], \quad (3.12 b)$$

where T = surface tension of the fluid.

(c) Dispersion relation

Since we have assumed that the quantity a is periodic in time we can re-write the eq. (2.5) as

$$-\rho a \omega^2 \cos \varphi = [-\nabla p + \mathbf{j} \times \mathbf{B} - \rho g \hat{\mathbf{z}}]_{zb}, \quad (3.13)$$

where the suffix zb on the right-hand side shows that the z -component of the quantities should be evaluated at the perturbed boundary. Using eqs. (3.2) to (3.8), (3.11) and (3.12) and including the effect of the homogeneous magnetic field produced by external sources, one can show, retaining terms of order a/d only, that eq. (3.13) reduces to

$$\omega^2 = \left[\frac{T k^3}{\rho} + g k + \frac{\mu_0 j_0^2 k_x^2 (2kd - 1)}{2\rho k^2} + \frac{k_x k_y}{\rho k} j_0 B_u \right] \tanh 2kd. \quad (3.14)$$

4. Discussion

The dispersion relation given by eq. (3.14) is a generalisation of the one obtained by Rayleigh [6] to the case when the fluid is an electrical conductor carrying a uniform current in the direction of a homogeneous magnetic field produced by external sources. We see that if we put $k_x = 0$ in the eq. (3.14)

Table 1. Values of the non-dimensional parameter $x=2kd$ for which the free surface of mercury in an infinitely large tank is neutral, x_c , and has a maximum growth rate, x_m , for sinusoidal perturbations of the boundary in the presence of uniform volume current. There is no external magnetic field. Effect of gravity and surface tension are included. The wave vector of the modes is directed perpendicular to the direction of the current.

j_0 ($A\text{ cm}^{-2}$)	$2d=0.1\text{ cm}$		$2d=0.2\text{ cm}$	
	x_c	x_m	x_c	x_m
316.2	0.045	0.023	0.086	0.044
632.5	0.150	0.076	0.262	0.134
948.7	0.255	0.135	0.420	0.216

the effect of both current and magnetic field is removed, but if we put $k_y=0$ in the eq. (3.14) the effect of the external magnetic field is removed and the effect of current is maximum because $k_x=k$ in this case. This is due to the fact that the fluid is assumed to have such low electrical conductivity that the magnetic field is not frozen inside. If $B_u=0$ and we also neglect surface tension and gravity eq. (3.14) shows that the fluid slab is stable for perturbations of wavelength $\lambda < \lambda_c$ and unstable when $\lambda > \lambda_c$ where

$$\lambda_c = 4\pi d. \quad (3.15)$$

Thus under these conditions the surface of the fluid is unstable when it carries a volume current and deforms at a maximum rate into ridges and valleys directed along the current. When surface tension and gravity are also included, the critical wavelength, λ_c , at which the surface is neutral to perturbation, is larger than that given by eq. (3.15).

If we define a non-dimensional parameter

$$x \equiv 2kd, \quad (3.16)$$

the value of $x=x_c$ at which the surface is neutral to perturbation and the value $x=x_m$ with $k_y=0$ at which it grows at a maximum rate when $B_u=0$ are calculated graphically for mercury ($T=487\text{ dyne cm}^{-1}$; $\rho=13.6\text{ gm cm}^{-3}$) for three values of current density and two values of depth and the results are shown in the Table 1.

Corresponding to $x=x_m$, the maximum growth rate of the unstable modes is calculated from eq. (3.14) when $B_u=0$ and the results are shown in Fig. 2, where it will be seen that the characteristic time of growth of instabilities is of the order of a hundred ms and that this quantity is not very sensitive to the depth of the fluid tank. The values of current density used in Fig. 2 are of the order of a few hundred amp cm^{-2} which corresponds to a power of order 10^4 watts for a tank of mercury with depth 0.1 cm, length 100 cm and width 30 cm.

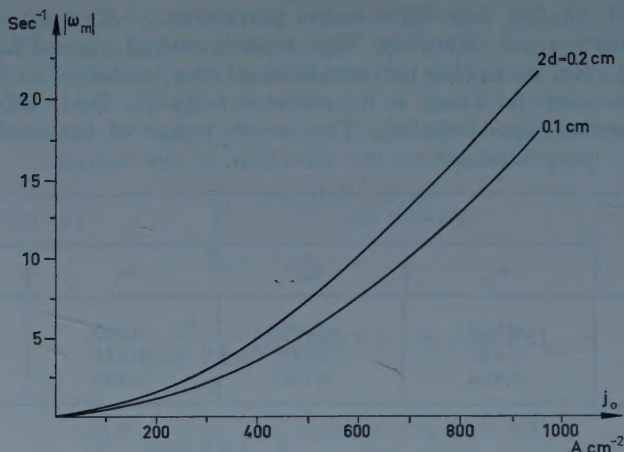


Fig. 2. Variation of the growth rate of infinitesimal sinusoidal perturbations of the free surface of a mercury tank with electric current density. The wave vector of the perturbation is perpendicular to the direction of the current. Effects of surface tension and gravity are included. The value of the parameter $2d$ is equal to the depth of the tank, the bottom of which is assumed to be in contact with a rigid surface.

If one puts T , g and B_u equal to zero in eq. (3.14) and considers the disturbance with $k_x = k$, one can show that the e -folding time of the disturbance is approximately equal to $4d/V_a$ where V_a is the Alfvén wave velocity corresponding to the magnetic field at the boundary of the slab. This result is in conformity with the observation made by Murty (§ 4 (b) of [4]) that the time scales defined by Alfvén wave propagation over characteristic distances of the problem play a significant role even in finitely conducting media if the electric currents are maintained by external sources.

If the value of $B_u \neq 0$, the behaviour of the fluid slab is more complicated. An examination of the dispersion relation shows that an external magnetic field directed along the current makes the fluid slab unstable when k_x and k_y have opposite signs. In such a case the wave vector of the perturbation lies in the second quadrant of the x - y -plane of the free surface and the resultant current density vector given by eq. (3.2) lies in the first quadrant of the x - y -plane at the points where the surface is perturbed upwards. This means that the wave vector of the unstable modes is so directed as to make the current density tend to be force free. This result is the same as the one obtained in the case of an infinite cylinder carrying axial current in the presence of a homogeneous magnetic field directed parallel to the current [4].

If the value of the external magnetic field, B_u , is so large that the effect of all other terms in eq. (3.14) can be neglected except the one containing B_u , we see that ω^2 has numerically the largest value when $|k_x| = |k_y| = k/\sqrt{2}$. Therefore, under these conditions the free surface is highly unstable for large wave-numbers of the perturbations with the wave vector lying in the second quadrant of the x - y -plane of the free surface inclined at an angle of 45° to the y -axis (direction of the current and the homogeneous magnetic field).

PART II

Instability of a fluid slab with both the surfaces free, carrying a uniform current in the presence of a homogeneous magnetic field**5. Formulation of the problem**

In this section we shall consider the stability of an incompressible, inviscid and nonmagnetic fluid slab with both surfaces free and carrying a uniform current in the presence of an external homogeneous magnetic field. The geometry of this problem is the same as the one shown Fig. 1, except for the difference that we shall not include the influence of gravity and assume that the surface $z = -d$ is also a free surface. This configuration can approximately be realised by a jet of rectangular cross-section, the thickness of which is small compared with the dimensions of the jet in the other directions. As before we assume that there is a uniform current given by $\mathbf{j} = (0, j_0, 0)$ flowing in the slab and that the current is closed symmetrically on either side of the slab at infinitely large distance. Also we introduce a uniform external magnetic field along the direction of the current.

When the two surfaces of the fluid slab are free there are in general three possible ways in which a sinusoidal perturbation can deform these surfaces. The deformed surfaces can be (1) symmetric or (2) anti-symmetric or (3) be neither symmetric nor anti-symmetric with respect to the plane $z = 0$. We shall show that only the cases (1) and (2) arise when $B_u = 0$ and only the case (3) arises when $B_u \neq 0$.

In order to do so we assume that the surface $z = +d$ (hereafter called upper surface) and the surface $z = -d$ (hereafter called lower surface) are respectively deformed to shapes given by

$$z = d + a \cos \varphi, \quad (5.1 a)$$

and

$$z = -d - \alpha a \cos \varphi, \quad (5.1 b)$$

where α is a nondimensional number. As before we assume that a is periodic in time (eq. (2.4)) and that $a \ll d$. We also demand that $\alpha a \ll d$. The solution gives ω and also the possible values of α which then determine the modes that are most unstable.

The basic equations that we use are again the set of eqs. (2.5) to (2.9) with eqs. (2.1) and (2.2) to describe the equilibrium conditions.

6. Solution for ω

The method of obtaining the value ω in the present case is similar to the one adopted in §3 in the case of one free surface, except for differences arising out of the boundary conditions. The general form of the distribution of current density, magnetic field and pressure inside the fluid are given by eqs. (3.2), (3.7) and (3.10) respectively. The constants in eq. (3.2) have now to be determined by demanding that there is no current density normal to the boundaries

given by eqs. (5.1). The constants in eq. (3.7) have to be determined by matching the interior solutions of the magnetic field with the exterior solutions at the two boundaries given by eq. (5.1). The constants in eq. (3.10) have to be determined by the condition that the pressure at a point just inside the boundaries is that due to surface tension only. Carrying out the above steps and using eq. (3.13) at both the upper and lower surfaces and retaining terms of order a/d or of order $\alpha a/d$, as the case may be, one can show that

$$\omega^2 = \frac{1}{\rho} [(1 + \alpha)(P + Q) + (1 - \alpha)(M + N)], \quad (6.1)$$

where
$$\alpha = \frac{1}{\xi} [-\eta \pm (\eta^2 - \xi\theta)^{\frac{1}{2}}], \quad (6.2)$$

where
$$\xi = P + Q - M - N, \quad (6.3 a)$$

$$\eta = Q + N, \quad (6.3 b)$$

$$\theta = Q + M - P - N, \quad (6.3 c)$$

where
$$P = \left[\frac{1}{2} T k^3 + \frac{1}{2 k^2} \mu_0 j_0^2 k_x^2 (kd - \cosh kd \exp(-kd)) \right] \tanh kd, \quad (6.4 a)$$

$$Q = \frac{1}{2k} j_0 B_u k_x k_y \coth kd, \quad (6.4 b)$$

$$M = \left[\frac{1}{2} T k^3 + \frac{1}{2 k^2} \mu_0 j_0^2 k_x^2 (kd - \sinh kd \exp(-kd)) \right] \coth kd, \quad (6.4 c)$$

and
$$N = \frac{1}{2k} j_0 B_u k_x k_y \tanh kd. \quad (6.4 d)$$

Eq. (6.2) shows that there are two possible values of α and correspondingly there are two possible values of ω^2 given by eq. (6.1), one for each value of α . The general form of the solution for ω is very involved as may be seen from eqs. (6.1) to (6.4). We therefore discuss the limiting cases when the external magnetic field is (1) zero, (2) very small and (3) very large, as these cases turn out to be particularly simple.

7. Limiting cases

(a) $B_u = 0$

If $B_u = 0$, we see from eqs. (6.4) that both Q and N are zero. Therefore η is also zero (eq. (6.3 b)) and $\xi = -\theta$ from eqs. (6.3). We then obtain from eq. (6.2) that under this limiting condition

$$\alpha = \pm 1. \quad (6.5)$$

In view of the general form of the perturbation we have started with (eq. (5.1)) we find that the fundamental modes are either symmetric ($\alpha=1$) or anti-symmetric ($\alpha=-1$). The corresponding dispersion relations are, for symmetric perturbation,

$$\omega^2 = \frac{2}{\varrho} P, \quad (6.6)$$

and for anti-symmetric perturbation,

$$\omega^2 = \frac{2}{\varrho} M. \quad (6.7)$$

It can be shown that $M \geq 0$ for all values of kd while P can be negative for some range of values of kd . Therefore we conclude that the fluid slab carrying a volume current in the absence of external magnetic field with both its surfaces free can be unstable for symmetric perturbations only. Further we note that in the presence of electric current the dispersion relation for symmetric deformation (eqs. (6.6) and (6.4 a)) is not identical to that for a fluid slab with half the thickness and with one surface rigid and the other remaining free (eq. (3.14) with $g=0$, $B_u=0$ and replacing $2d$ by d). This is due to the fact that the two sides of the plane $z=0$ in the present case are not independent but are in interaction due to the magnetic field due to the current flowing in the fluid.

$$(b) [B_u/\mu_0 j_0 d] \ll 1$$

If the external homogeneous magnetic field is small compared with the field produced by the current at the boundary of the slab, it can be shown from eqs. (6.2) to (6.4) that the two possible values of α , α_1 and α_2 , are given by

$$\alpha_1 = +1 + \beta_1 + O(\beta^2), \quad (6.8 a)$$

$$\alpha_2 = -1 + \beta_2 + O(\beta^2), \quad (6.8 b)$$

$$\text{where} \quad \beta_1 = \beta \coth kd, \quad (6.9 a)$$

$$\text{and} \quad \beta_2 = \beta \tanh kd, \quad (6.9 b)$$

$$\text{where} \quad \beta = \frac{j_0 B_u k_x k_y}{k(M-P)}. \quad (6.10)$$

Using eqs. (6.1), (6.8) and (6.9) one can show that up to the terms of first order in $B_u/\mu_0 j_0 d$ the dispersion relations are the same as those given by eqs. (6.6) and (6.7).

$$(c) [B_u/\mu_0 j_0 d] \gg 1$$

In the limit when the external homogeneous field is very large compared to the field at the boundary due to the current one can show from eqs. (6.2) to (6.4) that

$$\alpha = -\exp(\pm 2kd), \quad (6.11)$$

where the effect of surface tension is also neglected. It can be shown from eqs. (6.1) to (6.4) that in this limit of large external magnetic field the dispersion relation corresponding to the two values of α is given by

$$\omega^2 = \pm \frac{1}{\rho k} j_0 B_u k_x k_y, \quad (6.12)$$

according as

$$\alpha = -\exp(\pm 2kd).$$

In eq. (6.12) k_x and k_y may have either the same sign or have opposite signs. Therefore either of the values of α given by eq. (6.11) can correspond to unstable perturbations. The maximum instability occurs under this limiting condition when $|k_x| = |k_y| = k/\sqrt{2}$ and the wave vector is in the negative quadrant of the x - y -plane, and the growth rate is proportional to $(k)^{\frac{1}{2}}$.

We see from eqs. (6.8) and (6.11) that $\alpha \neq 1$ only when $B_u \neq 0$, which shows that the unstable modes of the fluid slab with both of its surfaces free are neither symmetric nor anti-symmetric with respect to the mid-plane in the presence of an external magnetic field.

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